Secondary 4 (Grade 10) – GEP Practice

2019 & 2020
Contest Problems with Full Solutions

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Section A (Correct answer – 2 points | No answer – 0 points | Incorrect answer – minus 1 point)

Question 1

Find the value of the following.

$$2020 \times 999 - 9999 \times 202 + 9999$$

A. 8181
B. 2020
C. –1818
D. 4047777
E. None of the above

Question 2

How many digits are there in $5^{2020} \times 4^{1008}$?

A. 2020
B. 1008
C. 2016
D. 2019
E. None of the above

Question 3

A standard 6-sided dice is rolled twice. Which of the following events has the highest probability of occurrence?

A. The second number is twice the first.
B. The second number is not greater than the first.
C. At least one number is greater than 3.
D. The sum of two numbers is a prime number.
E. None of the above

Question 4

Each of six countries brought a team of 5 players to a tennis tournament. Each pair of players from different countries played once. How many games were there in the tournament?

A. 250
B. 360
C. 375
D. 480
E. None of the above

Question 5

A "Nicely Seven" is a two-digit positive integer such that it has only two factors and the sum of the factors is a multiple of 7. How many Nicely Seven numbers are there?

A. 8
B. 13
C. 3
D. 4
E. None of the above

Question 6

Given that $a = 4^{\frac{1}{3}} - 2^{\frac{1}{3}} + 1$, what is the value of $\left(\frac{3-a}{a}\right)^6$?

A. 4
B. 2
C. 64
D. $\left(4^{\frac{1}{3}} - 2^{\frac{1}{3}}\right)^2$
E. None of the above

Question 7

Find the greatest prime number x which satisfies the following inequality.

$$\left|\frac{1}{57 - x}\right| < \frac{5}{x^2}$$

A. 13
B. 11
C. 59
D. 61
E. None of the above

Question 8

The number 4 can be expressed as the sum of the digits 1 or 2 in five different ways as shown below.

$4 = 1 + 1 + 1 + 1$

$4 = 1 + 1 + 2$

$4 = 1 + 2 + 1$

$4 = 2 + 1 + 1$

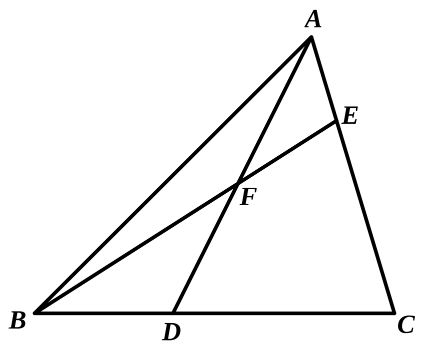
$4 = 2 + 2$

How many ways can the number 9 be expressed as the sum of the digits 1 or 2?

A. 60
B. 55
C. 45
D. 40
E. None of the above

Question 9

In the diagram, points E and D are on the sides AC and BC such that $AE = \frac{1}{3}AC$ and $BD = \frac{1}{3}BC$. What is the ratio of the area of triangle AEF to the area of triangle ABC?



A. $\frac{2}{7}$
B. $\frac{2}{9}$
C. $\frac{2}{13}$
D. $\frac{2}{15}$
E. None of the above

Question 10

Dennis wrote down all the 3-digit numbers in increasing order on a piece of paper. He used a red pen to write down the even numbers and a blue pen to write down the odd numbers. How many red digits '6' are there on the piece of paper?

A. 185
B. 195
C. 280
D. 900
E. None of the above

Question 11

It is given that a, b, and c are real numbers that satisfy the following equations.

$$\begin{cases} \dfrac{ab}{a+b} = 4 \\ \dfrac{bc}{b+c} = 6 \\ \dfrac{ac}{a+c} = 12 \end{cases}$$

What is the value of $\frac{abc}{ab+bc+ac}$?

A. 20
B. 2
C. 4
D. $\frac{1}{4}$
E. None of the above

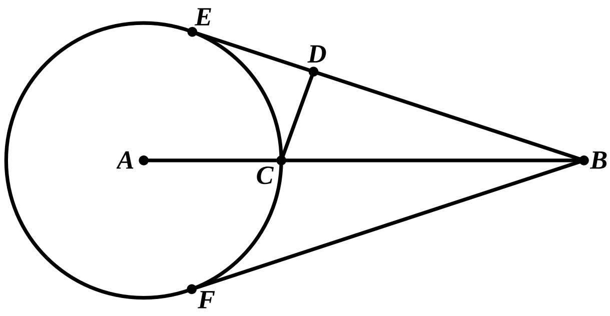
Question 12

How many pairs of two-digit positive integers a and b are there such that $a^2 + b^2 = 2020$?

A. 0
B. 1
C. 2
D. 4
E. None of the above

Question 13

In the figure below, the circle is centred at A. The segments BE and BF are tangents to the circle. Point C is on AB and on the circumference of the circle as shown. Point D is a point on BE such that CD is perpendicular to BE. If $BE = 12$ and $BC = 8$, find the length of CD.



A. $\frac{25}{13}$

B. $\frac{24}{13}$

C. $\frac{40}{13}$

D. $\frac{12}{5}$

E. None of the above

Question 14

Diana, Alexa, Jason, Tom and Michael are Secondary 2 or 4 students. They study in either Silver Oak Secondary School or Grand Mountain Secondary School. It is also given that:

- Tom and Michael are from different schools.
- Diana and Jason go to the same school.
- Three students go to Silver Oak Secondary School and the other two are from Grand Mountain Secondary School.
- Alexa and Michael are from the same grade.
- Jason and Tom study at different levels.
- Three students study in Secondary 2 and the other two students in Secondary 4.

If one of them is Secondary 4 student from Grand Mountain Secondary School, who is that person?

A. Diana
B. Alexa
C. Jason
D. Tom
E. Michael

Question 15

Thirty identical marbles are to be distributed to 6 boys. Each boy gets at least 3 marbles. How many ways are there to distribute the marbles to the boys?

A. 6188

B. 15625

C. 118755

D. $\frac{30!}{6}$

E. None of the above

Section B (Correct answer – 4 points| Incorrect or No answer – 0 points)

When an answer is a 1-digit number, shade "0" for the tens, hundreds and thousands place.
Example: if the answer is 7, then shade 0007
When an answer is a 2-digit number, shade "0" for the hundreds and thousands place.
Example: if the answer is 23, then shade 0023
When an answer is a 3-digit number, shade "0" for the thousands place.
Example: if the answer is 785, then shade 0785
When an answer is a 4-digit number, shade as it is.
Example: if the answer is 4196, then shade 4196

Question 16

Find the largest 4-digit multiple of 25 whose sum and product of its digits are both multiples of 25.

Question 17

What is the value of the following expression?

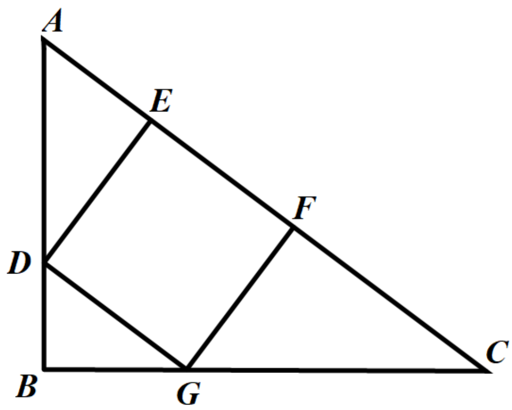
$$\frac{3\times 111.111}{6\times 1.001}+\frac{3\times 222.222}{6\times 2.002}+\frac{3\times 333.333}{6\times 3.003}+\cdots+\frac{3\times 666.666}{6\times 6.006}$$

Question 18

The six-digit number 21A3B8 is divisible by 33. Find the value of A + B.

Question 19

In the diagram, $\angle ABC = 90°$ and $DEFG$ is a square. If $AE = 8$ and $EF = 12$, find the area of quadrilateral $ACGD$.



Question 20

What is the greatest 2-digit number that can be written as the sum of 2 different prime numbers in exactly 2 different ways?

Question 21

What is the smallest positive integer that has exactly six odd divisors and 12 even divisors?

Question 22

If $(x^2 + 1)(y^2 + 1) + 9 = 6(x + y)$, find the value of $x^2 + y^2$.

Question 23

What is the least number of weights required to weigh any objects of integer number of grams from 1 to 35 grams? The weights must be in integer number of grams.

Question 24

In the following cryptarithm, all the different letters stand for different digits.

	C	I	R	C	L	E
	C	I	R	C	L	E
+	C	I	R	C	L	E
	S	P	H	E	R	E

Find the value of the sum C + I + R + C + L + E.

Question 25

Peter has seven different books of different subjects to be placed on a single-decked shelf. He does not want to place the Physics book next to the Biology one and the Geometry book next to the Chemistry one. In how many ways can he place all his books?

END OF PAPER

Solutions to SASMO 2020 Secondary 4 (Grade 10)

Question 1

$$2020 \times 999 - 9999 \times 202 + 9999 = 202 \times (9990 - 9999) + 9999 = 9999 - 202 \times 9$$
$$= \mathbf{8181}$$

Answer: **(A)**

Question 2

$$5^{2020} \times 4^{1008} = 5^{2020} \times (2^2)^{1008} = 5^{2020} \times 2^{2016} = 5^4 \times (5 \times 2)^{2016} = 625 \times 10^{2016}$$

625×10^{2016} has $3 + 2016 = \mathbf{2019}$ digits.

Answer: **(D)**

Question 3

Probability(an event) = (# of ways an event can happen) / (Total number of outcomes)

The total number of outcomes when a standard 6-sided dice is rolled twice is $6 \times 6 = 36$.

Find the probability for options A to D:

A. The number of ways option A can happen is 3: (1, 2), (2, 4) and (3, 6). P(A)=$3/36$.
B. The number of ways option B can happen is 21: (1, 1), (2, 1), (2, 2), (3, 1), …, (6, 6). P(B)=$21/36$.
C. The number of ways when both numbers are less than 4 is 9: (1, 1), (1, 2), (1, 3), (2, 1), …, (3, 3). P(C)=$1 - \frac{9}{36} = \frac{27}{36}$.
D. The number of ways option D can happen is 15: (1, 1), (1, 2), (1, 4), (1, 6), …, (6, 5). P(D)=$15/36$.

Option **C** has the highest probability of occurrence.

Answer: **(C)**

Question 4

There are $6 \times 5 = 30$ players in total. If all 30 players played with each other, then the total number of games would be $30 \times 29 \div 2 = 435$. However, the number of games between players from the same country is $5 \times 4 \div 2 = 10$. Thus, the number of games in the tournament was $435 - 6 \times 10 = \mathbf{375}$.

Answer: **(C)**

Question 5

A Nicely Seven number must be a prime number. Let a be a Nicely Seven number, then a and 1 are its factors and $a + 1$ is a multiple of 7.

List down all two-digit multiples of 7, then subtract 1 from them and check whether the results are prime numbers:

14	21	28	35	42	49	56	63	70	77	84	91	98
13	20	27	even	41	even	55	even	69	even	83	even	97

Thus, there are **4** Nicely Seven numbers.

Answer: **(D)**

Question 6

$$\left(2^{\frac{1}{3}}\right)^3 + 1^3 = \left(2^{\frac{1}{3}} + 1\right) \times \left(4^{\frac{1}{3}} - 2^{\frac{1}{3}} + 1\right) \Leftrightarrow 3 = \left(2^{\frac{1}{3}} + 1\right) \times a \Leftrightarrow$$

$$\frac{3}{a} - 1 = 2^{\frac{1}{3}} \quad \Leftrightarrow \frac{3-a}{a} = 2^{\frac{1}{3}} \Leftrightarrow \left(\frac{3-a}{a}\right)^6 = \left(2^{\frac{1}{3}}\right)^6 = \mathbf{4}$$

Answer: **(A)**

Question 7

Case 1: $x > 57$

$$\left|\frac{1}{57-x}\right| < \frac{5}{x^2} \Leftrightarrow \frac{1}{x-57} < \frac{5}{x^2} \Leftrightarrow x^2 - 5x + 285 < 0$$

The inequality $x^2 - 5x + 285 < 0$ does not have any solutions.

Case 2: $x < 57$

$$\left|\frac{1}{57-x}\right| < \frac{5}{x^2} \Leftrightarrow \frac{1}{57-x} < \frac{5}{x^2} \Leftrightarrow x^2 + 5x - 285 < 0$$

The solution of the inequality $x^2 + 5x - 285 < 0$ is

$$\frac{1}{2}\left(-5-\sqrt{1165}\right) < x < \frac{1}{2}\left(\sqrt{1165}-5\right).$$

The largest possible value of x is **13**.

Answer: **(A)**

Question 8

Count by the number of 2s in the sum as shown in the table below.

Number of 2s	Number of 1s	Number of ways
4	1	5
3	3	20
2	5	21
1	7	8
0	9	1
The total number of ways:		**55**

Answer: **(B)**

Question 9

Since $AE: EC = 1: 2$, then $Area(AEF): Area(CEF) = 1: 2$ and $Area(AEB): Area(CEB) = 1: 2$.

Similarly, $Area(BDA): Area(CDA) = 1: 2$ and $Area(BDF): Area(CDF) = 1: 2$.

Let $Area(AEF) = a$ and $Area(BDF) = b$, then $Area(CEF) = 2a$ and $Area(CDF) = 2b$.

From $Area(AEB): Area(CEB) = 1: 2$, we get $Area(AFB) = 1.5a$

From $Area(BDA): Area(CDA) = 1: 2$, we get $Area(AFB) = 1.5b$. Hence $a = b$, $Area(ABC) = 7.5a$ and $Area(AEF): Area(ABC) = a: 7.5a = \mathbf{\frac{2}{15}}$.

Answer: **(D)**

Question 10

Count the number of '6's in hundreds place:

When the hundreds place is 6, there are 10 options (0, 1, 2, …, 9) for tens place and 5 options (0, 2, 4, 6, 8) for ones place. There are $10 \times 5 = 50$ sixes in hundreds place.

Count the number of '6's in tens place:

When the tens place is 6, there are 9 options (1, 2, …, 9) for hundreds place and 5 options (0, 2, 4, 6, 8) for ones place. There are $9 \times 5 = 45$ sixes in tens place.

Count the number of '6's in ones place:

When the ones place is 6, there are 9 options (1, 2, …, 9) for hundreds place and 10 options (0, 1, 2, …, 9) for tens place. There are $9 \times 10 = 90$ sixes in tens place.

There are $50 + 45 + 90 = \mathbf{185}$ **red digits** '6' on the piece of paper.

Answer: **(A)**

Question 11

From the first equation:

$$\frac{ab}{a+b} = 4 \iff \frac{1}{4} = \frac{(a+b)}{ab} = \frac{1}{b} + \frac{1}{a}$$

Similarly, we get $\frac{1}{6} = \frac{1}{b} + \frac{1}{c}$ and $\frac{1}{12} = \frac{1}{c} + \frac{1}{a}$

Adding the 3 equations, we get

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2} \times \left(\frac{1}{4} + \frac{1}{6} + \frac{1}{12}\right) = \frac{1}{4}$$

Then

$$\frac{abc}{ab+bc+ac} = \frac{1}{\frac{ab+bc+ac}{abc}} = \frac{1}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} = \frac{1}{\frac{1}{4}} = \mathbf{4}$$

<u>Answer:</u> **(C)**

Question 12

If a and b are odd numbers, then a^2 and b^2 give remainder 1 on division by 4 and the sum $a^2 + b^2$ gives remainder 2 on division by 4. However, 2020 gives remainder 0 on division by 4. Hence a and b are even numbers.

Let $a = 2x$ and $b = 2y$ and rewrite the original equation:

$$(2x)^2 + (2y)^2 = 2020 \iff x^2 + y^2 = 505 \quad (1)$$

Obviously, x and y must have different parities.

If x is an even number, then only $x = 8$ and $x = 12$ satisfy the equation (1). Since the equation (1) is symmetric, then there are 4 solution pairs of (x, y) or **4 pairs** of (a, b): (42, 16), (38, 24), (24, 38) and (16, 42).

<u>Answer:</u> **(D)**

Question 13

Since the segment BE is tangent to the circle, then AE is perpendicular to BE and AEB and CDB are similar triangles.

Let $AE = AC = r$, then by Pythagoras Theorem in triangle AEB:

$$AE^2 + BE^2 = AB^2 \;\Leftrightarrow\; r^2 + 12^2 = (r+8)^2$$

Solving the equation, we get $r = 5$ and $AB = 13$.

From similar triangles AEB and CDB:

$$\frac{AE}{AB} = \frac{CD}{BC} \;\Leftrightarrow\; \frac{5}{13} = \frac{CD}{8} \;\Leftrightarrow\; CD = \mathbf{\frac{40}{13}}$$

Answer: **(C)**

Question 14

If Diana and Jason go to Grand Mountain Secondary School, then neither Tom nor Michael goes to Grand Mountain Secondary School. Hence Tom and Michael go to Silver Oak Secondary School which is impossible. Thus, Diana and Jason go to Silver Oak Secondary School.

If Alexa and Michael are Secondary 4 students, then neither Tom nor Jason is a Secondary 4 student. Hence Jason and Tom are Secondary 2 students which is impossible. Thus, Alexa and Michael are Secondary 2 students.

Diana, Jason, Alexa and Michael cannot be Secondary 4 student from Grand Mountain Secondary School. Thus, the answer is **Tom**.

Answer: **(D)**

Question 15

Let $a_1, a_2, \dots, a_6$ be the number of marbles given to each of the 6 boys. Then $a_1 \geq 3, a_2 \geq 3, \dots, a_6 \geq 3$ and rewrite $a_1 = b_1 + 2, a_2 = b_2 + 2, \dots, a_6 = b_6 + 2$ where $b_1 \geq 1, b_2 \geq 1, \dots, b_6 \geq 1$.

It is given that

$$(b_1 + 2) + (b_2 + 2) + \cdots (b_6 + 2) = 30 \quad \Leftrightarrow\; b_1 + b_2 + \cdots b_6 = 18$$

The number of solutions for the above equation is $C(17,5) = \binom{17}{5} = \frac{17!}{5! \times (17-5)!} =$ $\mathbf{6188}$.

Answer: **(A)**

Question 16

The largest possible sum of the digits of a 4-digit number is $9 \times 4 = 36$. Since the sum of digits is a multiple of 25, then the sum of the digits must be 25. The last 2 digits must be 00, 25, 50 or 75.

Case 1:

If the last 2 digits are 00, then the sum of the digits is at most 18 which is impossible.

Case 2:

If the last 2 digits are 25, then the sum of the hundreds and thousands digits is 18. However, $9 \times 9 \times 2 \times 5$ is not a multiple of 25.

Case 3:

If the last 2 digits are 50, then the sum of the digits is at most 23 which is impossible.

Case 4:

If the last 2 digits are 75, then the sum of the hundreds and thousands digits is 13. For the product to be a multiple of 25, the hundreds or thousands digit must be 5. Thus, the largest possible 4-digit number is **8575**.

Answer: **8575**

Question 17

$$\frac{3 \times 111.111}{6 \times 1.001} + \frac{3 \times 222.222}{6 \times 2.002} + \cdots + \frac{3 \times 666.666}{6 \times 6.006} =$$

$$\frac{3 \times 111 \times 1.001}{2 \times 3 \times 1.001} + \frac{3 \times 222 \times 1.001}{2 \times 3 \times 2.002} + \cdots + \frac{3 \times 666 \times 1.001}{2 \times 3 \times 6.006} =$$

$$\frac{111}{2} + \frac{111}{2} + \cdots + \frac{111}{2} = \frac{666}{2} = \mathbf{333}$$

Answer: **333**

Question 18

The six-digit number 21A3B8 is divisible by 3 and 11.

From the Divisibility Rule of 11:

$|(2 + A + B) - (1 + 3 + 8)| = |A + B - 10|$ must be divisible by 11. Hence $A + B = \mathbf{10}$.

Answer: **10**

Question 19

$$\angle ADE = 90° - \angle EAD = 90° - \angle CAB = \angle BCA = \angle GCF$$

$$\angle EAD = 90° - \angle ADE = 90° - \angle GCF = \angle FGC$$

Thus, triangles AED and GFC are similar and

$$\frac{AE}{DE} = \frac{GF}{FC} \quad \Leftrightarrow \quad \frac{8}{12} = \frac{12}{FC} \quad \Leftrightarrow \; FC = 18.$$

Quadrilateral $ACGD$ is trapezium and its area is

$$DE \times \frac{(GD + AC)}{2} = 12 \times \frac{(12 + 38)}{2} = \mathbf{300}.$$

Answer: **300**

Question 20

Clearly, the desired 2-digit number must be an even number. The numbers 98, 96, 94, …, 72, 70 can be written as the sum of 2 different prime numbers in more than 2 different ways. For example,

98	$79 + 19$	$31 + 67$	$37 + 61$
96	$7 + 89$	$13 + 83$	$17 + 79$
94	$5 + 89$	$11 + 83$	$23 + 71$
…			

The greatest 2-digit number that can be written as the sum of 2 different prime numbers in exactly 2 different ways is $\mathbf{68} = 31 + 37 = 7 + 61$.

Answer: **68**

Question 21

The number must have $6 + 12 = 18$ divisors. Then its prime factorisation must be $p^2 \times q^2 \times r,\ p \times q^8, p^2 \times q^5$or p^{17}. Since the number has exactly 6 odd divisors, then its prime factorisation must be $p^2 \times q^2 \times r$ or $p^2 \times q^5$. The smallest such number is $2^2 \times 3^2 \times 5 = \mathbf{180}$.

Answer: **180**

Question 22

$$x^2 + y^2 + x^2y^2 + 10 - 6x - 6y = 0 \Leftrightarrow$$

$$x^2 + y^2 + x^2y^2 + 10 - 6x - 6y + 2xy - 2xy = 0 \quad \Leftrightarrow$$

$$x^2y^2 - 2xy + 1 + x^2 + 2xy + y^2 - 6(x + y) + 9 = 0 \Leftrightarrow$$

$$(xy - 1)^2 + (x + y)^2 - 6(x + y) + 9 = 0 \Leftrightarrow$$

$$(xy - 1)^2 + [(x + y) - 3]^2 = 0$$

Hence $xy = 1,\ x + y = 3$ and

$$3^2 - 2 \times 1 = (x + y)^2 - 2xy = x^2 + y^2 = \mathbf{7}.$$

Answer: **7**

Question 23

To get the least number of weights, we need to put the weights on both sides of a scale.

To weigh 1 g object, we need 1 g weights.

To weigh 2 g object, we need 2 or 3 g weights.

To weigh 5 g and 14 g objects, we need at least 2 more weights. Thus, the least number of weights is **4** and the weights are 1 g, 3 g, 9 g and 27 g. For example,

5 g = 9 g – 3 g – 1 g, 14 g = 27 g – 9 g – 3 g – 1 g and 35 g = 27 g + 9 g – 1 g.

Answer: **4**

Question 24

Rewrite the sum as the product:

	C	I	R	C	L	E
×						3
	S	P	H	E	R	E

In ones place, the letter E must be 5. (E=0 can be checked similarly, and it doesn't lead to any solution.)

The six-digit number CIRCLE times 3 results in a 6-digit number, therefore C=1, 2 or 3.

If C=3, then in hundreds place product $C \times 3$, there must be a carry over of 6 from tens place product which is impossible.

If C=2, then in hundreds place product $C \times 3$, there must be a carry over of 9 from tens place product which is impossible. Hence C=1.

In hundreds place product $C \times 3$, there must be a carry over of 2 from tens place product which implies that L=8 or 9.

If L=8, then R=5=E which is impossible. Hence L=9, R=8 and H=4.

In hundred thousands place product, S cannot be 4 and 5 since H=4 and E=5. Hence S=3 and the only one-digit number left for I is 0.

The value of the sum C + I + R + C + L + E is $1 + 0 + 8 + 1 + 9 + 5 = \mathbf{24}$.

Answer: **24**

Question 25

The total number of ways to place all his books without any restrictions is $7! = 5040$.

The number of ways when the Physics book is next to the Biology book and Geometry is next to the Chemistry one is $5! \times 2 \times 2 = 480$.

Hence, the number of ways when the Physics book is not next to the Biology book and Geometry is not next to the Chemistry one is $5040 - 480 = \mathbf{4560}$.

Answer: **4560**