Secondary 3 (Grade 9) – GEP Practice

2020

Contest Problems with Full Solutions

Authors:

Henry Ong, BSc, MBA, CMA
Merlan Nagidulin, BSc

Section A (Correct answer – 2 points | No answer – 0 points | Incorrect answer – minus 1 point)

Question 1

Find the value of the following.

$$2020 \times 2020 - 2020 \times 20 + 2020$$

A. 2020
B. 4042020
C. 4040000
D. 4024020
E. None of the above

Question 2

Which of the following has the largest value?

A. 2020
B. $\frac{2020^{20}}{2020^{18}}$
C. 4080400
D. $\frac{2020^{20}-2020}{2020^{18}-2020}$
E. Options B and C

Question 3

In Mathematics, the product of the first n positive integers is written as $n! = n \times (n-1) \times \ldots \times 1$. For example, $2! = 2 \times 1$ and $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$. Find the number of consecutive zeros at the end of $(11! - 10!)$.

A. 1
B. 2
C. 24
D. 26
E. None of the above

Question 4

Find the value of the following expression.

$$\frac{2025 \times (19!)^2 - 5 \times (19!)^2}{100 \times 18! \times 20! + 18! \times 20!}$$

A. 19
B. 2020
C. 2025
D. $2020 \times 19!$
E. None of the above

Question 5

Which number below is a factor of $91^4 - 1$?

A. 2019
B. 2020
C. 2021
D. 2022
E. None of the above

Question 6

What is the smallest positive integer that can be expressed as a product of six positive integers such that any two of them are not coprime?

A. 60
B. 64
C. 360
D. 5040
E. None of the above

Question 7

Four friends have accounts in some social networks. Any two of them use at least one common social network, and no social networks are used by more than two of them. At least how many social networks are used by the friends?

A. 4
B. 5
C. 6
D. 8
E. None of the above

Question 8

The value of $f(x) = ax^2 + bx + c$ is 0 when $x = -1$. Given that $f(2) - f(1) = 11$ and $f(3) - f(2) = 15$, what is the value of $a + b + c$?

A. –3
B. –1
C. 1
D. 3
E. None of the above

Question 9

What is the next number in the sequence below?

$$16, 37, 58, 89, 145, 42, \ldots$$

A. 20
B. 31
C. 72
D. 96
E. None of the above

Question 10

There is one counterfeit coin among 2019 identical coins. The genuine coins have the same weight while the counterfeit coin is lighter. Andy wants to determine the counterfeit coin using a balance scale. There are no restrictions to the number of coins placed on the pans each time. What is the least number of weighings needed to find the counterfeit coin?

A. 7
B. 10
C. 11
D. 1009
E. None of the above

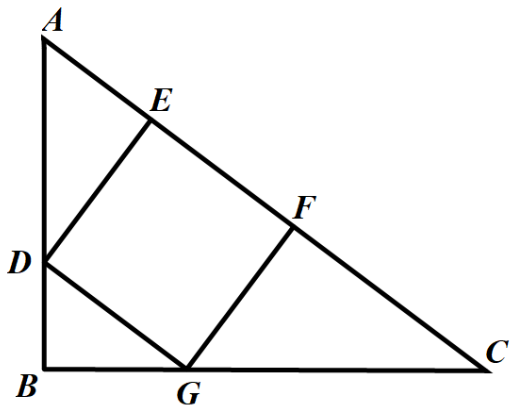
Question 11

How many four-digit numbers have digit 1 but not digit 2 among its digits?

A. 2248
B. 2673
C. 2816
D. 3528
E. None of the above

Question 12

In the diagram, $\angle ABC = 90°$ and $DEFG$ is a square. If $AE = 4$ and $EF = 6$, find the area of quadrilateral $ACGD$.



A. 84
B. 75
C. 63
D. 60
E. None of the above

Question 13

Anthony, Judith, Daniel and Edward were born in 1975, 1976, 1977 and 1978 in the cities of New York, Moscow, Barcelona and Singapore, though not in that order. It is given that

- The person born in Barcelona is one year older than the one born in Singapore.
- Edward was born one year later than the person born in Barcelona.
- Daniel was born in Moscow.
- Anthony was born two years later than Edward was.

Which year was Judith born in?

A. 1975
B. 1976
C. 1977
D. 1978
E. Impossible to determine

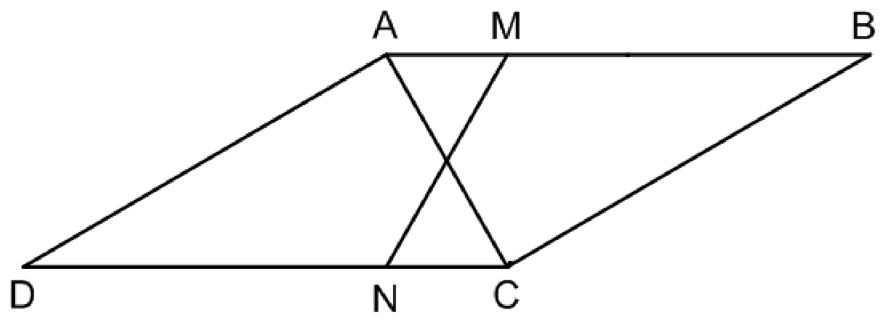
Question 14

A convex polyhedron has 48 edges. Its faces are either quadrilaterals or hexagons. Each vertex is an endpoint of exactly three edges. How many faces does the polyhedron have?

A. 16
B. 18
C. 20
D. 24
E. None of the above

Question 15

In the diagram below, $ABCD$ is a parallelogram and $\angle ACB = 90°$. Points M and N are on sides AB and CD such that $AM : MB = CN : ND = 1 : 3$. Given that $AC = MN = 2$, find the area of ABCD.



A. $2\sqrt{3}$
B. 6
C. $4\sqrt{3}$
D. 8
E. None of the above

Section B (Correct answer – 4 points| Incorrect or No answer – 0 points)

When an answer is a 1-digit number, shade "0" for the tens, hundreds and thousands place.
Example: if the answer is 7, then shade 0007
When an answer is a 2-digit number, shade "0" for the hundreds and thousands place.
Example: if the answer is 23, then shade 0023
When an answer is a 3-digit number, shade "0" for the thousands place.
Example: if the answer is 785, then shade 0785
When an answer is a 4-digit number, shade as it is.
Example: if the answer is 4196, then shade 4196

Question 16

Find the value of the following expression.

$$11^2 - 12^2 + 13^2 - 14^2 + 15^2 - 16^2 + \cdots + 99^2 - 100^2 + 101^2$$

Question 17

How many digits are there in $5^{2020} \times 4^{1010}$.

Question 18

The six-digit number 2X3Y72 is divisible by 66. Find the value of X + Y.

Question 19

Take a 2-digit number, multiply its digits, then multiply the digits of the obtained product and continue the process until you get a one-digit number. How many two-digit numbers will produce zero at the end of this process?

Question 20

Find the value of $(a+2b)(2b+3c)(a+3c)+6abc$ where $a+2b+3c=0$.

Question 21

Given that $P(x)=x^5-2021x^4+2021x^3-2021x^2+2021x-1$, what is the value of $P(2020)$?

Question 22

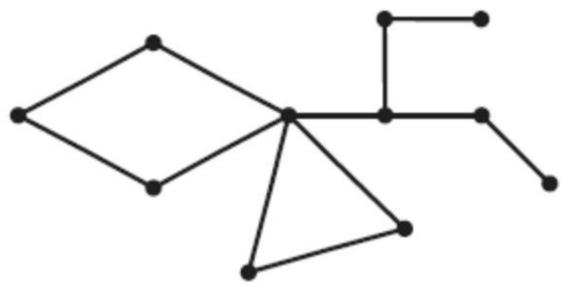
In the following cryptarithm, all the different letters stand for different digits.

	C	I	R	C	L	E
	C	I	R	C	L	E
+	C	I	R	C	L	E
	S	P	H	E	R	E

Find the value of the sum S + P + H + E + R + E.

Question 23

In how many ways can the vertices in the diagram below be coloured with 3 colours such that any two neighbouring vertices are not coloured with the same colour?

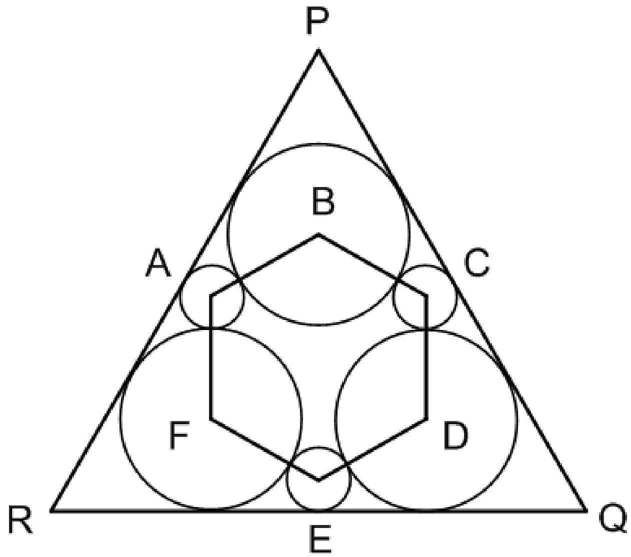


Question 24

Let $T_1 = 1$, $T_2 = 1 + \frac{1}{1}$, $T_3 = 1 + \frac{1}{1+\frac{1}{1}}$, ... , $T_n = 1 + \frac{1}{T_{n-1}}$. It is given that T_n can be expressed as a fraction in its simplest form, $T_n = \frac{P_n}{Q_n}$, where P_n and Q_n are positive coprime integers. How many numbers among P_1, P_2, P_3, ... , P_{2019} are odd?

Question 25

In the diagram below, there are 6 touching circles with centres in A, B, C, D, E and F. It is given that ABCDEF is a regular hexagon and AB = 4. PQR is a triangle such that each of its sides touches three of the circles. Find the square of the length of PQ.



END OF PAPER

Solutions to SASMO 2020 Secondary 3 (Grade 9)

Question 1

$$2020 \times 2020 - 2020 \times 20 + 2020 = 2020 \times (2020 - 20 + 1) = 2020 \times 2001$$
$$= \mathbf{4042020}$$

Answer: **(B)**

Question 2

A. 2020

B. $\frac{2020^{20}}{2020^{18}} = 2020^2 = 4080400$

C. 4080400

D. $\frac{2020^{20}-2020}{2020^{18}-2020} = \frac{2020^{19}-1}{2020^{17}-1} = \frac{2020^{19}-2020^2+2020^2-1}{2020^{17}-1} = \frac{2020^{19}-2020^2}{2020^{17}-1} + \frac{2020^2-1}{2020^{17}-1}$

$$= \frac{2020^2(2020^{17}-1)}{2020^{17}-1} + \frac{2020^2-1}{2020^{17}-1} = 2020^2 + \frac{2020^2-1}{2020^{17}-1}$$

Option D has the largest value.

Answer: **(D)**

Question 3

$11! - 10! = 10!\,(11-1) = 10 \times 10!$

$10!$ contains 2 pairs of (2,5) in its product , hence 10! has two consecutive ending zeros and $10 \times 10!$ has **3** consecutive zeros**.**

Answer: **(E)**

Question 4

$$\frac{2025 \times (19!)^2 - 5 \times (19!)^2}{100 \times 18! \times 20! + 18! \times 20!} = \frac{(19!)^2 \times (2025-5)}{18! \times 20! \times (100+1)} = \frac{(18! \times 19) \times 19! \times 2020}{18! \times (19! \times 20) \times 101}$$
$$= \frac{19 \times 2020}{20 \times 101} = \mathbf{19}$$

Answer: **(A)**

Question 5

$$91^4 - 1 = (91^2 - 1)(91^2 + 1) = 8280 \times 8282 = 20 \times 414 \times 101 \times 82$$
$$= 20 \times 101 \times 414 \times 82 = \mathbf{2020} \times 414 \times 82$$

Answer: **(B)**

Question 6

1 cannot be one of the six positive integers as 1 is coprime to all positive integers. Then each of the six positive integers must be greater than 1. Thus, the smallest positive desired integer is $\mathbf{64} = 2 \times 2 \times 2 \times 2 \times 2 \times 2$.

Answer: **(B)**

Question 7

As each pair of friends requires a separate social network, the result cannot be less than $\binom{4}{2} = 6$. It's possible to construct a solution with exactly 6 social networks. For example, if we call the friends as A, B, C, D and number the social networks by numbers from 1 to **6**, then the solution can be {A: (1, 2, 3), B: (1, 4, 5), C: (2, 4, 6), D: (3, 5, 6)}.

Answer: **(C)**

Question 8

It is given that

$$f(-1) = a - b + c = 0 \qquad (1)$$
$$11 = f(2) - f(1) = (4a + 2b + c) - (a + b + c) = 3a + b \qquad (2)$$
$$15 = f(3) - f(2) = (9a + 3b + c) - (4a + 2b + c) = 5a + b \qquad (3)$$

Solving equations (2) and (3), we get $a = 2$ and $b = 5$. From equation (1), $c = b - a = 5 - 2 = 3$. The value of $a + b + c$ is $2 + 5 + 3 = \mathbf{10}$.

Answer: **(E)**

Question 9

Each number is the sum of the squares of digits of the previous number:

$37 = 1^2 + 6^2$

$58 = 3^2 + 7^2$

$89 = 5^2 + 8^2$

$145 = 8^2 + 9^2$

$42 = 1^2 + 4^2 + 5^2$

The next number in the sequence is $4^2 + 2^2 = \mathbf{20}$.

<u>Answer:</u> **(A)**

Question 10

This is an example of a well-known balancing puzzle. It can be proved that for n coins the solution is $\lceil \log_3 n \rceil$, using a divide-and-conquer strategy: put one third of the coins on one pan and the same number of coins on the other pan. Therefore, the answer is **7**.

<u>Answer:</u> **(A)**

Question 11

The total number of 4-digit numbers is 9000:

Thousands place	Hundreds place	Tens place	Ones place	Total
9 options (1, 2, 3, 4, …, 9)	10 options (0, 1, 2, 3, 4, …, 9)	10 options (0, 1, 2, 3, 4, …, 9)	10 options (0, 1, 2, 3, 4, …, 9)	$9 \times 10 \times 10 \times 10 = 9000$

The number of 4-digit numbers with no digit '2's is 5832:

Thousands place	Hundreds place	Tens place	Ones place	Total
8 options (1, 3, 4, …, 9)	9 options (0, 1, 3, 4, …, 9)	9 options (0, 1, 3, 4, …, 9)	9 options (0, 1, 3, 4, …, 9)	$8 \times 9 \times 9 \times 9 = 5832$

Hence, the number of 4-digit numbers which contain digit '2' is $9000 - 5832 = 3168$.

The number of 4-digit numbers which do not contain digit '1' and '2' is 3584:

Thousands place	Hundreds place	Tens place	Ones place	Total
7 options (3, 4, …, 9)	8 options (0, 3, 4, …, 9)	8 options (0, 3, 4, …, 9)	8 options (0, 3, 4, …, 9)	$7 \times 8 \times 8 \times 8 = 3584$

Hence, the number of 4-digit numbers which contain digit '1' or '2' is $9000 - 3584 = 5416$.

Thus, the number of four-digit numbers which have digit 1 but not digit 2 among its digits is $5416 - 3168 = \mathbf{2248}$.

Answer: **(A)**

Question 12

$$\angle ADE = 90^\circ - \angle EAD = 90^\circ - \angle CAB = \angle BCA = \angle GCF$$

$$\angle EAD = 90^\circ - \angle ADE = 90^\circ - \angle GCF = \angle FGC$$

Thus, triangles AED and GFC are similar and

$$\frac{AE}{DE} = \frac{GF}{FC} \iff \frac{4}{6} = \frac{6}{FC} \iff FC = 9.$$

Quadrilateral $ACGD$ is trapezium and its area is

$$DE \times \frac{(GD + AC)}{2} = 6 \times \frac{(6 + 19)}{2} = \mathbf{75}.$$

Answer: **(B)**

Question 13

From the first two statements, we get that the person born in Barcelona is one year older than Edward who was born in Singapore.

From the 4th statement, Anthony is two years younger than Edward (from Singapore) and three years younger than the person born in Barcelona. Hence Anthony was born in 1978 in New York since Daniel was born in Moscow.

Judith was born in **1975** in Barcelona.

Answer: **(A)**

Question 14

As every vertex belongs to three edges, and every edge connects two vertices, the total number of vertices is equal to $48 \times 2 \div 3 = 32$.

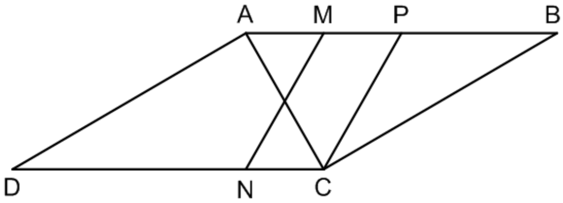
Since the polyhedron is convex and therefore simply connected, we can use Euler's formula. Using Euler's Formula, $F + V = E + 2$, where F is the number of faces, V is the number of vertices, and E is the number of edges, we get

$$F = E - V + 2 = 48 - 32 + 2 = \mathbf{18}.$$

Answer: **(B)**

Question 15

Let us draw segment CP parallel to MN where P lies on AB. Since $MP = NC = AM = \frac{1}{4}AB$, then $AP = PB = \frac{1}{2}AB$. Since $CP = MN = AC$ and $\angle ACB$ is a right angle, then triangle ABC is a right-angled triangle where the length of one of its legs is equal to the half the length of its hypotenuse. Thus, triangle ABC has angles 60, 30 and 90 degrees. Therefore, the length of BC is $\sqrt{3} \times AC = 2\sqrt{3}$, and the area of ABCD is $2 \times \frac{1}{2} \times 2 \times 2\sqrt{3} = \mathbf{4\sqrt{3}}$.



Answer: **(C)**

Question 16

$$11^2 - 12^2 + 13^2 - 14^2 + 15^2 - 16^2 + \cdots + 99^2 - 100^2 + 101^2 =$$

$$= (11 - 12) \times (11 + 12) + (13 - 14) \times (13 + 14) + \cdots + (99 - 100) \times (99 + 100) + 101^2 = -1 \times (11 + 12 + \cdots + 99 + 100) + 101^2 =$$

$$= -1 \times \left(\frac{100 \times 101}{2} - \frac{10 \times 11}{2}\right) + 101^2 = 101^2 - 50 \times 101 - \big(-(5 \times 11)\big) =$$

$$= 101 \times (101 - 50) + 55 = \mathbf{5206}$$

Answer: **5206**

Question 17

$$5^{2020} \times 4^{1010} = 5^{2020} \times (2^2)^{1010} = 5^{2020} \times 2^{2020} = (5 \times 2)^{2020} = 10^{2020}$$

10^{2020} has **2021** digits.

Answer: **2021**

Question 18

The six-digit number 2X3Y72 must be divisible by 2, 3 and 11.

From the Divisibility Rule of 11:

$|(2+3+7)-(X+Y+2)| = |10-X-Y| = 0$ or $X+Y=\mathbf{10}$.

Answer: **10**

Question 19

First, we list down all 2-digit numbers that produce zero after one operation.

After 1 operation: 10, 20, 30, 40, 50, 60, 70, 80, 90 – 9 numbers

Next, we list down the 2-digit numbers that will produce the 2-digit numbers listed above.

25, 52, 45, 54, 56, 65, 58, 85 – 8 numbers

Finally, we list down the 2-digit numbers which will produce the 2-digit number in the previous list of 8 numbers.

55, 59, 95, 96, 69, 87, 78 – 7 numbers.

The process is stopped here as no two-digit numbers can produce any of the 7 numbers in the last list.

Thus, there are $9+8+7=\mathbf{24}$ such two-digit numbers.

Answer: **24**

Question 20

From the equation $a+2b+3c=0$, we get $(a+2b)=-3c, (2b+3c)=-a$ and $(a+3c)=-2b$. Thus,

$$(a+2b)(2b+3c)(a+3c)+6abc = (-3c)\times(-a)\times(-2b)+6abc = -6abc+6abc = \mathbf{0}$$

Answer: **0**

Question 21

$$\begin{aligned}P(x) &= x^5 - 2021x^4 + 2021x^3 - 2021x^2 + 2021x - 1\\ &= x^5 - 2020x^4 - x^4 + 2020x^3 + x^3 - 2020x^2 - x^2 + 2020x + x - 1\\ &= x^4(x - 2020) - x^3(x - 2020) + x^2(x - 2020) - x(x - 2020) + x - 1\end{aligned}$$

Thus,

$$P(2020) = 2020 - 1 = \mathbf{2019}$$

Answer: **2019**

Question 22

Rewrite the sum as the product:

	C	I	R	C	L	E
×						3
	S	P	H	E	R	E

In ones place, the letter E must be 5. (E=0 can be checked similarly, and it doesn't lead to any solution.)

The six-digit number CIRCLE times 3 results in a 6-digit number, therefore C=1, 2 or 3.

If C=3, then in hundreds place product $C \times 3$, there must be a carry over of 6 from tens place product which is impossible.

If C=2, then in hundreds place product $C \times 3$, there must be a carry over of 9 from tens place product which is impossible. Hence C=1.

In hundreds place product $C \times 3$, there must be a carry over of 2 from tens place product which implies that L=8 or 9.

If L=8, then R=5=E which is impossible. Hence L=9, R=8 and H=4.

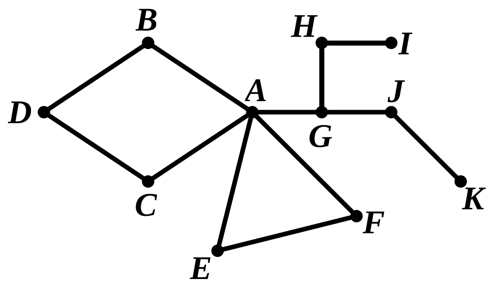
In hundred thousands place product, S cannot be 4 and 5 since H=4 and E=5. Hence S=3 and the only one-digit number left for I is 0.

The value of the sum S + P + H + E + R + E is $3 + 2 + 4 + 5 + 8 + 5 = \mathbf{27}$.

Answer: **27**

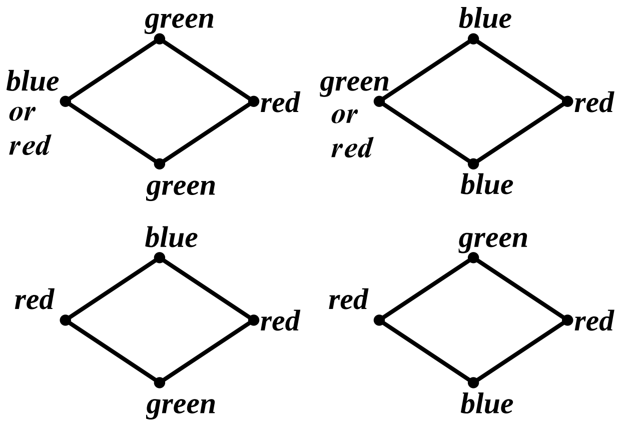
Question 23

Label the vertices as shown below.

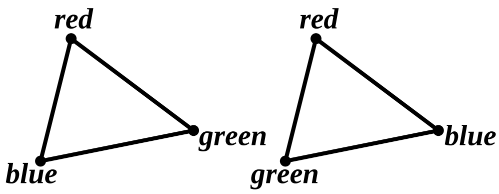


Let us colour point A with red colour.

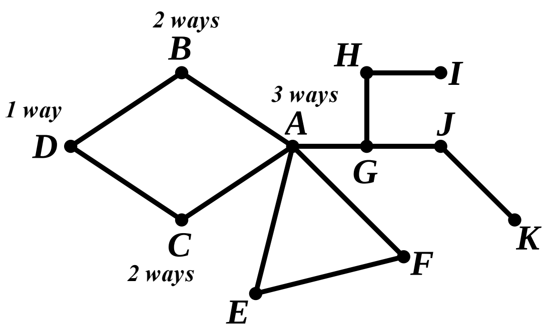
There are 6 ways to colour quadrilateral ABDC.



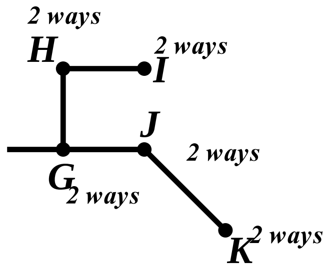
There are 2 ways to colour triangle GEF.



Point A – 3 ways, points B and C – 2 ways each since one colour was used on point A and point D – 1 way since two colours were used on points B and C.



There are $2^5 = 32$ ways to colour the GHIJK.



Thus, when point A is red, there are $6 \times 2 \times 32 = 384$ ways. Similarly, there are 384 ways each to colour the entire figure when point A is blue or green.

Hence, the total number of ways to colour the figure is $384 \times 3 = \mathbf{1152}$.

Answer: **1152**

Question 24

First, calculate some T_n values when n is small:

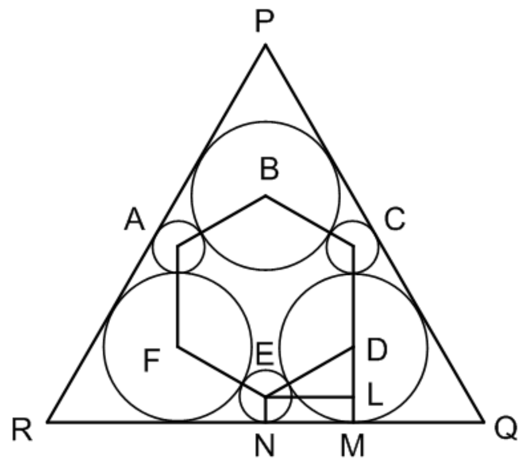
$$T_1 = \frac{1}{1}, T_2 = \frac{2}{1}, T_3 = \frac{3}{2}, T_4 = \frac{5}{3}, T_5 = \frac{8}{5}.$$

Note that both numerators and denominators correspond to Fibonacci numbers.
Using mathematical induction, it can easily be proved that $P_n = P_{n-1} + P_{n-2}$ and $Q_n = P_{n-1}$ for every $n \geq 3$.
We can notice that $P_1 = odd$, $P_2 = even$, $P_3 = odd$, $P_4 = odd$, $P_5 = even$, $P_6 = odd$ and so on where the sequence $odd, even, odd$ will keep repeating. Thus, there are $2019 \div 3 \times 2 = \mathbf{1346}$ odd numbers among P_1, P_2, P_3, ..., P_{2019}.

Answer: **1346**

Question 25

Draw lines DM and EN perpendicular to Q, where M and N lie on QR, and line EL parallel to QR as shown.



Since $ABCDEF$ is a regular hexagon, then PQR is also a regular triangle and CD is perpendicular to QR. Therefore C, D and M lie on a straight line, $\angle EDL = 180° - \angle CDE = 60°$ and $DL = \frac{ED}{2}$.

Since $EN + DM = ED$, we have

$$DM = ED - EN \Rightarrow$$

$DL = DM - LM = (ED - EN) - EN = ED - 2EN \Rightarrow \frac{ED}{2} = ED - 2EN \quad \Rightarrow \quad EN = \frac{ED}{4} = \frac{AB}{4} = 1$

Also, $NM = EL = \frac{\sqrt{3}}{2} \times ED = 2\sqrt{3}$ and $MQ = \sqrt{3}DM = 3\sqrt{3}$. Therefore, $QR = PQ ==$ $2 \times (2\sqrt{3} + 3\sqrt{3}) = 10\sqrt{3}$, and the square of its length is $(10\sqrt{3})^2 = \mathbf{300}$.

Answer: **300**